SOME MATHEMATICAL ASPECTS OF OPTIMAL PREDATION IN ECOLOGY AND BOVICULTURE

Richard Bellman
Mathematics Division
Robert Kalaba
Engineering Division
The RAND Corporation

P-1911

17 February 1960
Revised 8 March 1960

Approved for OTS release

Reproduced by

The RAND Corporation • Santa Monica • California

The views expressed in this paper are not necessarily those of the Corporation

CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION, CFSTI
INPUT SECTION 410.11

AD 616565

LIMITATIONS IN REPRODUCTION QUALITY OF TECHNICAL ABSTRACT BULLETIN DOCUMENTS, DEFENSE DOCUMENTATION CENTER (DDC)

- [ ] 1. AVAILABLE ONLY FOR REFERENCE USE AT DDC FIELD SERVICES. COPY IS NOT AVAILABLE FOR PUBLIC SALE.

- [x] 2. AVAILABLE COPY WILL NOT PERMIT FULLY LEGIBLE REPRODUCTION. REPRODUCTION WILL BE MADE IF REQUESTED BY USERS OF DDC.

- [x] A. COPY IS AVAILABLE FOR PUBLIC SALE.

- [ ] B. COPY IS NOT AVAILABLE FOR PUBLIC SALE.

- [ ] 3. LIMITED NUMBER OF COPIES CONTAINING COLOR OTHER THAN BLACK AND WHITE ARE AVAILABLE UNTIL STOCK IS EXHAUSTED. REPRODUCTIONS WILL BE MADE IN BLACK AND WHITE ONLY.

TSL-121-2/65

DATE PROCESSED: 6-28-65

PROCESSOR: [illegible]

SUMMARY

General mathematical problems arising in the scientific study of predation have been studied from a variety of viewpoints. Primary emphasis has been given to the descriptive aspects of the prey and predator populations under various assumptions concerning interactions among the different members of the populations during these processes and to birth and death processes. In particular, we wish to call attention to the recent paper by MacArthur, as well as to the classical works of Volterra, Lotka, and Chiang.

The major objective of this note is to show how the functional equation technique of a new mathematical discipline, dynamic programming, can be used in formulating and solving—both analytically and numerically—a variety of problems of optimal predation. We wish to determine optimal predation policies and are thus interested in the control, as opposed to the descriptive, aspects of predation processes.

SOME MATHEMATICAL ASPECTS OF OPTIMAL PREDATION IN ECOLOGY AND BOVICULTURE

Richard Bellman and Robert Kalaba
The RAND Corporation
Santa Monica, California

1. Introduction

General mathematical problems arising in the scientific study of predation have been studied from a variety of viewpoints. Primary emphasis has been given to the descriptive aspects of the prey and predator populations under various assumptions concerning interactions among the different members of the populations during these processes and to birth and death processes. In particular, we wish to call attention to the recent paper by MacArthur [1], where additional references can be found, as well as to the classical works of Volterra [2], Lotka [3], and Chiang, [6].

The major objective of this note is to show how the functional equation technique of a new mathematical discipline, dynamic programming [4] can be used in formulating and solving--both analytically and numerically--a variety of problems of optimal predation. We wish to determine optimal predation policies and are thus interested in the control, as opposed to the descriptive, aspects of predation processes.

2. An Ecological Process

Let us suppose that members of two different populations, called type I and type II, are present, and that members of type I prey on members of type II, but not conversely. Let us furthermore suppose that the presence of neither type is desirable (e.g. bacterial populations), and that we possess a drug (or other technique such as radiation) which is effective in destroying members of population type I, but is ineffective against those of type II. Our task is to determine the most efficacious mode of administration of the drug in an effort to control the two populations. The problem is not trivial since the administration of maximal dosages of the "drug" may deplete the type I population to such an extent that type II population may become dangerously large.

To cast such a problem in more mathematical form let us introduce a bit of nomenclature:

(1) $u(t)$ = size of population I at time t,

(2) $v(t)$ = size of population II at time t,

(3) $w(t)$ = rate of administration of the drug at time t.

We then consider the populations to be described by the differential equations and initial conditions

$$(4)\qquad \begin{aligned} \dot{u} &= k_1(w(t))u, \quad u(0) = c_1, \\ \dot{v} &= k_2(w(t))v - k_3(w(t))u, \quad v(0) = c_2, \end{aligned}$$

which hold for $u, v \geq 0$ and for $0 \leq t \leq T$, where T is the duration of the process. We measure the undesirability of the two populations at any time t by the expression $a_1u(t) + a_2v(t)$. Our objective is to determine $w = w(t)$, $0 \leq t \leq T$, in such a manner that we make the functional

$$(5)\qquad \max_{0 \leq t \leq T} \{a_1u(t) + a_2v(t)\}$$

be as small as possible. In place of the usual forcing term effect of control, we assume in writing (4), that the influence of the drug is to affect the growth and interactions of the two populations.

We introduce the function $f(c_1,c_2,T)$, defined by the relationship

$$(6)\qquad f(c_1,c_2,T) = \min_{w} \max_{0 \leq t \leq T} \{a_1u(t) + a_2v(t)\},$$

and use the principle of optimality [4] to derive the functional equation

$$(7)\qquad f(c_1,c_2,T) = \max\Big[a_1c_1 + a_2c_2, \min_{W} f(c_1 + k_1(W)c_1h, c_2 + k_2(W)c_2h - k_3(W)c_1h, T - h)\Big] + o(h).$$

This can be used for computational purposes, in conjunction with the terminal condition

$$f(c_1,c_2,0) = a_1c_1 + a_2c_2, \tag{8}$$

or can be used as the basis for further analytical studies. Lastly, we remark that the function $f(c_1,c_2,T)$ is homogeneous of degree one in c_1 and c_2, a fact which can be used to advantage computationally to reduce the dimension of f from two to one.

3. Boviculture

Suppose we have a herd of cattle with x_i head of age i, $i = 0,1,2,\ldots,K-1$, and x_K head of age K or older. Each year we can either send some of the cattle to market, each individual of age i being of worth w_i, or we can keep them to build up the herd. We suppose that x_i members of age i give rise to b_ix_i calves in a year and that a_i is the fraction of individuals of age i surviving to age $i+1$. For an N-stage process, $N = 1,2,\ldots,$ we wish to find the optimal breeding policy, that is, the policy that results in maximal overall return. We shall assume that the last decision must be to send all cattle to market.

We introduce the sequence of functions

(1) $f_N(x_0,x_1,x_2,\ldots,x_K)$ = the return from an N-stage process beginning with x_i cattle of age i and using an optimal predation policy, $N = 1,2,\ldots.$

First we observe that

$$f_1(x_0,x_1,x_2,\ldots,x_K) = \sum_{i=0}^{K} w_ix_i, \tag{2}$$

and then we use the principle of optimality to derive the functional relationships

$$f_{k+1}(x_0,x_1,\ldots,x_K) = \max_{\substack{0\le y_j\le x_j\\ j=0,1,2,\ldots,K}} \left\{ \sum_{i=0}^{K} w_jy_j + f_k\left(\sum_{j=0}^{K} b_j(x_j-y_j), a_0(x_0-y_0), \ldots, a_{N-2}x_{N-2}, a_{N-1}x_{N-1} + a_Nx_N\right)\right\}, \tag{3}$$

valid for $k = 1,2,\ldots$.

The sequence of functions $f_1, f_2, \ldots$ is determined recursively, beginning with f_1. Simultaneously, we determine the optimal values of y_1, which tell us how many cows in each age group to send to market in terms of the current composition of the herd and the time remaining in the process.

Problems of this nature can be solved analytically in a number of cases; cf. Bellman, Glicksberg, Gross [7].

4. Discussion

The two simplified models which have been sketched here serve but to suggest what can be done with regard to predation and breeding processes using modern mathematical techniques and employing high-speed digital computing machines. Additional papers, in which a variety of stochastic and adaptive [5] features are incorporated, will be prepared.

We wish to acknowledge the fruitful discussions that we have had with I. Cooper, J. Digby, J. Jacquez, and J. Lyman.

REFERENCES

1. R. MacArthur, "On the relation between reproductive value and optimal predation," Proc. Nat. Acad. Sci. USA, vol. 46, 1960, pp. 143-145.

2. V. Volterra, Theory of Functionals and of Integral and Integro-differential Equations, Dover Publ. Inc., New York (reprinted).

3. A. Lotka, Elements of Mathematical Biology, Dover Publ. Inc., New York (reprinted).

4. R. Bellman, Dynamic Programming, Princeton Univ. Press, Princeton, N. J., 1957.

5. R. Bellman and R. Kalaba, "A mathematical theory of adaptive control processes," Proc. Nat. Acad. Sci. USA, vol. 45, 1959, pp. 1288-1290.

6. C. Chiang, "Competition and other interactions between species," Statistics and Mathematics in Biology, O. Kempthorne, T. Bancroft, J. Gowen, and J. Lush (eds.), The Iowa State College Press, Ames, Iowa, 1954.

7. R. Bellman, I. Glicksberg, and O. Gross, Some Aspects of the Mathematical Theory of Control Processes, The RAND Corporation, Report R-313, 1958.